# Solutions - Assignmet #6

Determine the order of the pole at $z=0$

#1 (a) $f(z) = \dfrac{\cosh z}{z^3}$ - since $\cosh z \neq 0$, $z=0$ is a pole of order 3.

(b) $f(z) = \dfrac{e^{4z}-1}{\sin^2 z} = \dfrac{g(z)}{h(z)}$

$g(z) = e^{4z} - 1 = 0$ at $z=0$

$g'(z) = 4e^{4z} \neq 0$

so $z=0$ is a single zero of $g(z)$

$h(z) = \sin^2 z$ $\quad h(0) = 0$

$h'(z) = 2\sin z \cos z$ $\quad h'(0) = 0$

$h''(z) = -2\sin^2 z + 2\cos^2 z$ $\quad h''(0) \neq 0$

$z=0$ is a zero of order 2 for $h(z)$

so $f(z) = \dfrac{g(z)}{h(z)} = \dfrac{(z-0)\,\tilde{g}(z)}{(z-0)^2\,\tilde{h}(z)}$ where $\tilde{g}$ & $\tilde{h}$ are analytic and non-zero at $z=0$

$= \dfrac{1}{z-0}\,\dfrac{\tilde{g}}{\tilde{h}}$ So $z=0$ is a pole of order 1 for $f(z)$

#2 Find and classify the singularities

(a) $f(z) = \dfrac{1}{z(e^z-1)} = \dfrac{1}{g(z)}$ $\quad g(z) = z(e^z-1) = 0$ at $z=0$ & $e^z = 1$ at $z = 2n\pi i$ $\quad n = 0, \pm 1, \pm 2, \ldots$

| | $z=0$ | $z=2n\pi i$, $n\neq 0$ |
|---|---|---|
| $g(z) = z(e^z-1)$ | $0$ | $0$ |
| $g'(z) = (e^z - 1 + ze^z)$ | $0$ | $\neq 0$ |
| $g''(z) = e^z + e^z + ze^z = 2e^z + ze^z = e^z(2+z$ | $\neq 0$ | |

So $z=0$ is a pole of order 2

$z = 2\pi i n$, $n \neq 0$ is a single pole.

(b) $g(z) = \dfrac{\sin z}{z^2 - z}$

$\left.\begin{array}{l}\sin z = 0 \text{ at } z=0 \\ \cos z \neq 0 \text{ at } z = 0\end{array}\right\} \Rightarrow$ $z=0$ is a single zero of $\sin z$.

| | $z=0$ | $z=1$ |
|---|---|---|
| $z^2 - z$ | $0$ | $0$ |
| $\frac{d}{dz} \to 2z-1$ | $\neq 0$ | $\neq 0$. |

$z=0$ & $z=1$ are single zeros of $z^2-z$.

$g(z) = \dfrac{z\, h(z)}{z(z-1)} = \dfrac{h(z)}{(z-1)}$ is analytic at $z=0$.

so $z=0$ is a removable singularity.

since $\sin z$ at $z=1$ is non-zero, $z=1$ is a simple pole for $g(z)$

(c) $h(z) = \dfrac{\tan z}{z}$ since $\tan z = 0$ and $\frac{d}{dz}(\tan z) = \sec^2 z \neq 0$ at $z=0$.

$= \dfrac{z\, \tilde{h}(z)}{z} = \tilde{h}(z)$ is analytic

so $z=0$ is a removable singularity.

#3. Let $C = \{z \mid |z| = 5\}$ with positive orientation.

Evaluate

(a)
$$\oint_{|z|=5} \sin\left(\frac{1}{z}\right) dz = 2\pi i \operatorname{Res}(f;0)$$

Since there is only one singular point inside $C$.

we compute the residue

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

so

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots$$

so $\operatorname{Res}(f;0) = 1 \Rightarrow \oint_C \sin\left(\frac{1}{z}\right) dz = 2\pi i$.

(b)
$$\oint_C z \sin\frac{1}{z}\, dz = 2\pi i \operatorname{Res}(f;0) = 0$$

$$z \sin\frac{1}{z} = 1 - \frac{1}{3!z^2} + \frac{1}{5!z^4} - \cdots$$

so $\operatorname{Res}(f;0) = 0$

(c)
$$\oint_C z^2 \sin\frac{1}{z}\, dz = 2\pi i \operatorname{Res}(f;0) = -\frac{2\pi i}{3!} = -\frac{\pi i}{3}$$

$$z^2 \sin\frac{1}{z} = z - \frac{1}{3!} + \frac{1}{5!z^3} - \cdots$$

$$\operatorname{Res} f(;0) = -\frac{1}{3!}$$

#4. Compute Res$(f;0)$

(a) $f(z) = z^2 e^{1/z}$

we have $z^2 e^{1/z} = z^2 \left\{ 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots \right\}$

$= z^2 + z + \frac{1}{2!} + \frac{1}{3!z} + \cdots$

$\text{Res}(f;0) = \frac{1}{3!} = \frac{1}{6}$.

(b) $f(z) = \frac{1+e^z}{1-e^z}$ $\Rightarrow$ $\text{Res}(f;0) = \lim_{z\to 0} \frac{z(1+e^z)}{1-e^z}$

$z=0$ is a single pole.

$\overset{H}{=} \frac{(1+e^z) + ze^z}{-e^z} = -2$

(c) $f(z) = \frac{e^z - 1}{\sin z}$

$z=0$ is a single zero

$z=0$ is a single zero

$z=0$ is a removable singularity $\Rightarrow$ no poles.

so $\text{Res}(f;0) = 0$