Threshold Dominating Cliques in Random Graphs and Interval Routing ⋆

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Abstract

The existence of (shortest-path) interval routing schemes for random graphs that use at most one interval label per edge is an open problem posed in [8]. In this paper, we show that for any random graph $G(n, p)$ with edge probability $p > 0.765$, there exists an interval routing scheme that uses at most one label per edge and has an additive stretch 1. In doing so, we provide an interesting construction of such an interval routing scheme for graphs that have a $\frac{1}{2}$-threshold dominating clique, and establish a general result on the existence of threshold dominating cliques in random graphs.

Key words: Interval routing schemes, threshold dominating cliques, random graphs

1 Introduction

Routing is one of the most important tasks in distributed systems and interconnected networks. A routing scheme specifies how messages are delivered in a network. In a routing scheme, each node is associated with a routing table that specifies for each destination, the outgoing link through which messages to the destination should be forwarded.

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An *interval routing scheme* is a compact way to represent a routing table. In an interval routing scheme, the vertices in a network are labeled by a set of integers $\{1, 2, \cdots, n\}$. For each vertex, each outgoing link is labeled by zero or more intervals of integers. At any given vertex, messages to a given destination v are forwarded along the unique outgoing link labeled by an interval that contains the label of the destination v. A good routing scheme should assign as few as possible intervals to the outgoing links without sacrificing much on the length of the routing path — ideally, messages should be sent along the shortest path to their destination.

Studying the complexity (lower and upper bounds) and designing efficient algorithms for constructing interval routing schemes have attracted much interest [5,7,8,11]. In this paper, we show that for random graphs $G(n, p)$ with $p > 0.765$, there exists with high probability an interval routing scheme that uses at most one label per edge and has an additive stretch 1. In doing so, we provide an interesting construction of such an interval routing scheme on graphs that have a $\frac{1}{2}$-threshold dominating clique, and prove a general result on the existence of threshold dominating cliques in random graphs.

Our proof can be modified to show that every random graph $G(n, p)$ with $p > \frac{3-\sqrt{5}}{2} \approx 0.387$ has a standard dominating clique, thereby providing an immediate and significant improvement to the lower bound $p > 0.5$ on the existence of an interval routing scheme that uses at most one label per edge and has an additive stretch 2 ([11]).

The rest of this paper is organized as follows. In the next section, we discuss related work. In Section 3, we present the main results. In Section 4, we detail our construction of the interval routing scheme. In Section 5, we prove theorems on the existence of threshold dominating cliques in random graphs. We conclude in Section 5 with some discussions on the complexity of finding threshold dominating cliques.

2 Previous Work

There has been much interest in studying the complexity (lower and upper bounds) and designing efficient algorithms for constructing interval routing schemes [5,7,8,11]. Interval routing schemes for random graphs have also been investigated intensively. In [5], lower and upper bounds on the minimum number of labels per edge in a shortest-path interval scheme are established for random graphs $G(n, p)$ where $\frac{\log^{\epsilon} n}{n} < p < n^{-\frac{1}{2}-\epsilon}$ for some constant $\epsilon > 0$.

In [7,8], the existence of shortest-path interval routing schemes for random graphs $G(n, p)$, where $0 < p < 1$ is a fixed constant, are studied in three different settings. These settings differ in how the labels of the vertices are assigned: randomly-assigned, adversary-assigned, and designer-assigned. The current work deals with the case of designer-assigned routing schemes, i.e., the designer of the routing schemes has the power to specify the labels of the network nodes. In the rest of this paper, when we talk about the design of an interval routing scheme, we always assume that the designer is to specify the labeling of the vertices.

Gavoille and Peleg proved in [8] that for any random graph $G(n, p)$ with $p \geq \frac{1}{2}$ there exists, with high probability, a shortest-path interval routing scheme that uses at most 2 intervals per edge. In fact, their interval routing scheme is constructed in such a way that for each vertex, all but at most $O(\log^3 n)$ outgoing edges are labeled by one interval. Gavoille and Peleg [8] posed the open question on the existence of shortest-path interval routing schemes that use at most one interval per edge in random graphs.

In [11], it is shown that for any random graph $G(n, p)$ with $p \geq \frac{1}{2}$ there exists, with high probability, an interval routing scheme that uses at most one label per edge and has an additive stretch 2. The construction is based on the existence of a dominating clique, and we believe that an additive stretch 2 is the best possibility to achieve by using their approach.

We note that for any constant edge probability $0 < p < 1$, the diameter of $G(n, p)$ can be shown to be 2 with high probability. This applies to the work in [8,11] as well as the current work, but doesn't indicate that the problem is trivial. On the other hand, it is known (see, e.g., [7]) that for any graph with diameter 2, an interval routing scheme exists that uses at most one interval per edge and induces routing paths of length at most 4. To see this, recall that the height of a breadth-first search tree is at most 2 for any graph of diameter 2. Such a spanning-tree-based interval routing scheme, however, may have a routing path with additive stretch 3. The work in [8] provides an improvement by showing that a shortest-path interval routing scheme exists that uses at most two intervals per edge; The work in [11] provides an improvement by showing that an interval routing scheme exists that uses one interval per edge but has an additive stretch 2.

3 Main Results

Our main result is that for any random graph $G(n, p)$ with $p > 0.765$ there exists, with high probability, an interval routing scheme that uses at most one label per edge and has an additive stretch 1. To establish the result, we show that such an interval routing scheme can be constructed for any graph that has a $\frac{1}{2}$-threshold dominating clique. We further establish a general result on the existence of threshold dominating cliques in random graphs, which are interesting in their own right.

Throughout the paper, a network is modeled by an undirected graph $G(V, E)$. For a vertex $v \in V$, we use $N(v)$ to denote the set of neighbors of v, i.e.,

$$N(v) = \{u \in V | u \neq v \text{ and } (u, v) \in E\}.$$

We use $|\cdot|$ to denote the cardinality of a set.

Definition 3.1 *An α-threshold dominating clique of a graph $G(V, E)$ is a subset V_D of vertices such that*

(1) V_D induces a clique, and
(2) each vertex $v \in V \setminus V_D$ has more than $\alpha|V_D|$ neighbors in V_D, i.e.,

$$|N(v) \cap V_D| > \alpha|V_D|.$$

First we have the following

Theorem 1 *If a graph $G(V, E)$ has a $\frac{1}{2}$-threshold dominating clique, then there exists an interval routing scheme for $G(V, E)$ that uses at most one label per edge and has an additive stretch 1.*

We then prove a general result on the existence of α-threshold dominating cliques in the random graph $G(n, p)$ for any $\alpha > 0$. The result, together with Theorem 1, implies that for any random graph $G(n, p)$ with $p > 0.765$, there exists an interval routing scheme that uses at most one label per edge and has an additive stretch 1.

Theorem 2 *Let $G(n, p)$ be a random graph and $0 < \alpha < 1$ be any integer. The probability that $G(n, p)$ has an α-threshold dominating clique satisfies*

$$\begin{aligned}
&\lim_n \mathbb{P}\left\{G(n, p) \textit{ has an } \alpha\textit{-threshold dominating clique }\right\} \\
&\qquad = \begin{cases} 1, \textit{ if } p > p_\alpha, \\ 0, \textit{ if } p < \alpha, \end{cases} \qquad (1)
\end{aligned}$$

where p_α is the unique solution in the interval $[\alpha, 1]$ to the equation

$$4(p - \alpha)^2 \log_{\frac{1}{p}} e = 1. \qquad (2)$$

For $\alpha = \frac{1}{2}$, $p_\alpha \approx 0.765$.

Applying Theorem 1 and Theorem 2, we get

Corollary 3.1 *For any random graph $G(n,p)$ with $p > 0.765$, there exists with high probability an interval routing scheme that uses at most one label per edge and has an additive stretch 1.*

Our proof on the existence of threshold dominating cliques can be modified to deal with the case where we only require that any vertex is dominated by more than a fixed number of vertices in the clique.

Let $\beta > 0$ be a constant. Let p_β be the probability that the random graph $G(n,p)$ has a clique V_β such that every vertex $v \notin V_\beta$ has more than β neighbors in V_β.

We have the following result on the probability p_β, which was first reported by the author in the conference paper [2] for the case of $\beta = 1$. The case $\beta = 1$ already provides an immediate and significant improvement to the previous lower bound $p > 0.5$ established in [11] on the existence of an interval routing scheme that uses at most one label per edge and has an additive stretch 2.

Theorem 3 *For any random graph $G(n,p)$ and any $\beta > 0$, we have*

$$\lim_n p_\beta = \begin{cases} 1, \text{ if } p > \frac{3-\sqrt{5}}{2} \approx 0.387, \\ 0, \text{ if } p < \frac{3-\sqrt{5}}{2}. \end{cases}$$

4 The Interval Routing Scheme

In this section, we prove Theorem 1 by presenting a concrete interval routing scheme. Throughout this section, let $G(V,E)$ be a graph on n vertices and V_D be a $\frac{1}{2}$-threshold dominating clique of size $|V_D| = r$.

An *interval routing scheme* consists of a vertex-labeling function $\ell(v) : V \to \{1, 2, \cdots, n\}$ and an edge-labeling function $L(u,v)$ defined on $V \times V$.

For each ordered pair of vertices u and v such that (u,v) is an edge, the edge-labeling function $L(\cdot)$ assigns a collection $L(u,v)$ of zero or more intervals for the outgoing link from u to v and a collection $L(v,u)$ of zero or more intervals for the outgoing link from v to u. In a network, $L(v,u)$ is to be implemented at the node v and $L(u,v)$ is to be implemented at the node u. In this sense, the edge-labeling function treats the graph as directed even though the underlying graph model is undirected.

Given two integers $a \leq b$, we will be using $[a,b]$ to denote the interval of integers $[a, a+1, \cdots, b]$. A *singleton interval* is an interval that contains only one integer.

In order for a pair of vertex-labeling function and edge-labeling function to be an interval routing scheme, they need to satisfy the following properties.

Definition 4.1 (Interval Routing Scheme, [7]) *Let $\ell(\cdot)$ be a vertex-labeling function and $L(\cdot,\cdot)$ be an edge-labeling function defined over the vertex set of a graph $G(V,E)$. The pair $(\ell(\cdot), L(\cdot,\cdot))$ is an interval routing scheme if the following conditions are satisfied:*

(1) The vertex-labeling function is one-to-one.
(2) For each $v \in V$, the union of the intervals of the edge labels on v and $\ell(v)$ is $\{1, 2, \cdots, n\}$, i.e.,

$$\{L(v,w) | (v,w) \in E\} \cup \{\ell(v)\} = \{1, 2, \cdots, n\}.$$

(3) For any $v \in V$ and any other two vertices $u, w \in V$,

$$L(v,u) \cap L(v,w) = \emptyset.$$

(4) For any $u, v \in V$, there exists a sequence of vertices $(u_0, \cdots, u_m)$ such that $u_0 = u$, $u_m = v$, and for any $1 \le i \le m$,

$$\ell(v) \in L(u_{i-1}, u_i).$$

A message from a source vertex u to a destination v is forwarded along the unique path described in the last condition of the above definition. At any given vertex of the path, the message is forwarded along the unique outgoing link labeled by an interval that contains the label $\ell(v)$ of the destination.

4.1 The Vertex-Labeling Function

To design the vertex-labeling function, we partition the vertex set V into groups with special structures by making use of the $\frac{1}{2}$-threshold dominating clique V_D of $G(V,E)$. Each group, except for the last one, contains a unique vertex in V_D and a subset of its neighbors in $V \setminus V_D$. The last group contains at least $\lceil \frac{1}{2} r \rceil$ vertices in V_D (recall that $r = |V_D|$), each of which is adjacent to every other vertex in the last group. This last group serves as a bridge to handle messages that are otherwise hard to route with an additive stretch 1. The existence of this group of bridge vertices distinguishes our construction from the previous clique-based vertex-labeling schemes in the literature [8,11]. The following Lemma describes the partition and guarantees its existence.

Lemma 4.1 *There is a partition $\mathcal{P} = \{V_1, V_2, \cdots, V_k\}$ of the vertex set V into $|\mathcal{P}| = k \le \lceil \frac{1}{2} r \rceil$ groups that satisfy the following conditions:*

(1) For any $1 \le i \le k-1$,

$$V_i = \{u_i\} \cup U_i$$

where $u_i \in V_D$ *is a distinguished vertex in* V_i, *and* $U_i \subset N(u_i) \cap (V \setminus V_D)$,

(2) For any pair of indices $1 \le i < j \le k-1$, *no vertex in* U_j *neighbors* u_i;

(3) The last group

$$V_k = \{u_k, \cdots, u_r\} \cup U_k$$

contains all the remaining vertices, where

$$\{u_k, \cdots, u_r\} = V_D \setminus \{u_1, \cdots, u_{k-1}\},$$

such that every vertex $u \in \{u_k, \cdots, u_r\}$ *is adjacent to every vertex in* U_k.

Proof. We show that the following simple algorithm (Algorithm 1) constructs such a partition. It iteratively constructs the groups V_1 through V_k. Once a group is formed, the vertices in that group are removed from the graph. In each iteration, the algorithm selects a vertex in V_D with the minimum number of neighbors in the remaining part of the graph, and forms a group that contains the selected vertex and its neighbors in the remaining part of the graph.

Algorithm 1 Vertex Partition

```
i = 1.
while V_D ≠ ∅ do
  Select a vertex u_i ∈ V_D with the minimum number of neighbors in V \ V_D.
  if u_i neighbors all the vertices in V \ V_D then
    V_i = V.
    return {V_1, ··· , V_i}.
  else
    V_i = {u_i} ∪ (N(u_i) ∩ (V \ V_D)).
    V = V \ V_i.
    V_D = V_D \ {u_i}.
    i = i + 1.
```

Let $\{V_1, \cdots, V_k\}$ be the partition constructed by the above algorithm. From the construction process, for each $1 \le i \le k-1$, the group V_i contains only one vertex from V_D. Since the vertices of a group will be removed from the graph, we see that for any $1 \le i < j \le k$, no vertex in U_j is adjacent to $u_i \in V_i$ — otherwise the vertex in U_j that is adjacent to u_i should have been included in the group V_i. Therefore, the first two conditions are satisfied.

Recall that $r = |V_D|$ is the size of the $\frac{1}{2}$-threshold dominating clique. In addition to all the vertices in $V \setminus V_D$ that remain, the last group V_k contains all the vertices in $V_D \setminus \{u_1, u_2, \cdots, u_{k-1}\}$. Denote these vertices by $\{u_k, \cdots, u_r\}$ so that

$$V_k = \{u_k, \cdots, u_r\} \cup U_k.$$

Since in each iteration, the algorithm selects a vertex in V_D with the minimum number of neighbors, we see that each $u_i, i \ge k$, is adjacent to every vertex in

$V_k \setminus V_D$. Therefore, the partition satisfies the third condition.

We now show that $k \leq \lceil \frac{1}{2} r \rceil$. Assume on the contrary that $k > \lceil \frac{1}{2} r \rceil$. Consider a vertex v in the last group V_k. By the construction process, this vertex v is not adjacent to any of the vertices in $\{u_1, \cdots, u_{k-1}\}$. If $k > \lceil \frac{1}{2} r \rceil$, we see that v cannot have more than $\lceil \frac{1}{2} r \rceil$ neighbors in V_D. A contradiction to the fact that V_D is a $\frac{1}{2}$-threshold dominating clique. ■

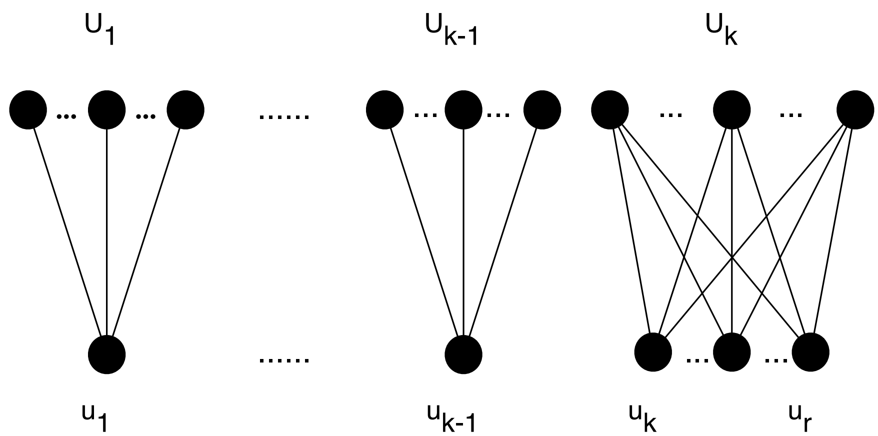

Fig. 1. An illustration of a partition of the vertex set. The vertices in the $\frac{1}{2}$-threshold dominating clique are at the bottom row and vertices in U_i's on the top.

In the following, we describe the vertex-labeling function $\ell(v)$. Let $\mathcal{P} = \{V_1, V_2, \cdots, V_k\}$ be the partition of V as described in Lemma 4.1 and illustrated in Figure 1. We label the vertices in a group of the partition $\mathcal{P}$ consecutively. For each group $V_i = \{u_i\} \cup U_i, i \leq k-1$, the vertex u_i is labeled first. For the last group V_k, the order of the labels of the vertices in V_D does not matter, but $u_i, \cdots, u_r$ will be labeled before the vertices in U_k.

Formally, let d_i be the size of V_i. Define $s_1 = 0$ and

$$s_i = d_1 + d_2 + \cdots + d_{i-1}, \quad \forall \;\; 1 < i \leq k.$$

For each group $V_i = \{u_i\} \cup U_i, i < k$, we label u_i by $s_i + 1$, i.e.,

$$\ell(u_i) = s_i + 1$$

and label the vertices in U_i by the integers $s_i + 2$ through $s_i + d_i$.

Vertices in the last group V_k are labeled in the following way. Recall that the subset of vertices $\{u_k, \cdots, u_r\}$ in V_D is contained in V_k. We label the vertices $\{u_k, \cdots, u_r\}$ by the integers $s_k + 1$ through $s_k + (r-k) + 1$, i.e.,

$$\ell(u_i) = s_k + (i-k) + 1, \quad \forall k \leq i \leq r.$$

and label the vertices in U_k by the integers $s_k + (r-k) + 2$ through n.

4.2 The Edge-Labeling Function

Let $\mathcal{P} = \{V_1, V_2, \cdots, V_k\}$ be the partition given in Lemma 4.1 together with the vertex-labeling function $\ell(v)$ specified in Section 4.1. Recall that $d_i = |V_i|$ is the size of the group V_i. Edges incident to the vertices in V_D and in $V \setminus V_D$ are labeled differently.

4.2.1 Labeling the Edges Incident to Vertices in V_D

Let $1 \leq i \leq k-1$. Consider the vertex $u_i \in V_D$ in the group $V_i = \{u_i\} \cup U_i$. There are three types of labels for routing messages from u_i to different destinations:

(1) **Labels for Messages within $\mathbf{V_i = \{u_i\} \cup U_i}$.**
For each vertex w in U_i, we label the outgoing link on the edge (u_i, w) by the singleton interval $[\ell(w)]$, i.e.,

$$L(u_i, w) = [\ell(w)].$$

(2) **Labels for Messages to Other Groups $\mathbf{V_j, j \leq k-1}$.**
For $1 \leq j \leq k-1, j \neq i$, we label the outgoing link on the edge (u_i, u_j) by the interval $[\ell(u_j), \ell(u_j) + d_j]$, i.e.,

$$L(u_i, u_j) = [\ell(u_j), \ell(u_j) + d_j] = [s_j + 1, s_j + d_j].$$

(3) **Labels for Messages to the Last Group $\mathbf{V_k}$.**
For each $k \leq j \leq r-1$, we label the outgoing link on the edge (u_i, u_j) by the singleton $[\ell(u_j)]$. We label the outgoing link on the edge (u_i, u_r) by the interval $[\ell(u_r), n]$, i.e.,

$$L(u_i, u_r) = [\ell(u_r), n],$$

which handles all the messages from u_i to the vertices in U_k.

For the vertices $\{u_k, \cdots, u_r\}$ in the last group V_k, the first two types of labels described in the above are used. Note that in this case, we also have the edge label $L(u_i, u_j) = [u_j]$ for every pair $(u_i, u_j) \subset \{u_k, \cdots, u_r\}$.

4.2.2 Labeling Edges Incident to the Vertices in $V \setminus V_D$

Let $1 \leq i \leq k$ and consider the group $V_i = \{u_i\} \cup U_i$ of the partition. Let u be a vertex in U_i.

For messages from $u \in U_i$ to vertices in the same group V_i, where $1 \le i \le k-1$, we use the following labeling scheme. We label the outgoing link on the edge (u, u_i) by the interval $[\ell(u_i), \ell(u_i) + d_i]$ to take care of the messages from u to vertices in V_i.

For messages to other groups V_j of the partition or messages from $u \in U_k$ to vertices in V_k, we consider two cases. We emphasize that the following discussions apply to any vertex $u \in U_i, 1 \le i \le k$. In particular, Case **B** is able to handle messages from $u \in U_k$ to the other vertices in V_k.

A. Vertex u is adjacent to at least one vertex in $\mathbf{V_j}$ and $\mathbf{j < k}$

The idea is based on that in [8] for routing messages from a vertex to a set of vertices that form a clique. In our case, using this technique is one of the main reasons that the obtained interval routing scheme has an additive stretch 1.

If w is the only vertex in V_j that is adjacent to u, we label the outgoing link on the edge (u, w) by the interval $[\ell(u_j), \ell(u_j) + d_j]$. Otherwise, let $\{w_1, w_2, \cdots, w_m\}$ be the subset of vertices in V_j that are adjacent to v and assume that they have the vertex labels

$$\ell(w_1) < \ell(w_2) < \cdots < \ell(w_m).$$

We label the edges involving v and w_i's as follows:

$$\begin{aligned} L(u, w_1) &= [\ell(u_j), \ell(w_2) - 1] \\ L(u, w_t) &= [\ell(w_t), \ell(w_{t+1}) - 1], \forall 1 < t < m \\ L(u, v_m) &= [\ell(w_m), \ell(u_j) + d_j] \end{aligned}$$

B. Vertex u is not adjacent to any vertex in $\mathbf{V_j}$ or $\mathbf{j = k}$

We need to handle this situation by considering all such V_j's simultaneously. Assume that there are m such "bad" groups $\{V_{i_1}, V_{i_2}, \cdots, V_{i_m}\}$ where $i_1 < i_2 < \cdots < i_m < k$ (we also regard the last group V_k as "bad"). The idea is to use one **dedicated** vertex from $\{u_k, \cdots, u_r\} = V_k \cap V_D$ as a bridge for each "bad" group.

Recall that $V_k = \{u_k, \cdots, u_r\} \cup U_k$ where $\{u_k, \cdots, u_r\} \subset V_D$. The following observation indicates that we can always do this because V_D is a $\frac{1}{2}$-threshold dominating clique.

Lemma 4.2 *If there are m "bad" groups for the vertex u, then u is adjacent to at least $m+1$ vertices in $\{u_k, u_{k+1}, \cdots, u_r\}$.*

Proof. Since V_D is a $\frac{1}{2}$-threshold dominating clique, u is adjacent to at least

$\lceil \frac{1}{2}r \rceil$ vertices in V_D. Recall that $k \leq \lceil \frac{1}{2}r \rceil$. Therefore the number of vertices in $\{u_k, u_{k+1}, \cdots, u_r\}$ that are adjacent to u is at least $\lceil \frac{1}{2}r \rceil - ((\lceil \frac{1}{2}r \rceil - 1) - m) = m + 1$. ■

Based on Lemma 4.2, we may assume that u is adjacent to a subset of $m+1$ vertices $\{u_{j_1}, u_{j_2}, \cdots, u_{j_{m+1}}\}$ in $V_k \cap V_D$, where $j_1 \geq k$. For each $1 \leq p \leq m$, we label the edge (u, u_{j_p}) by the interval $[\ell(u_{i_p}), \ell(u_{i_p}) + d_{i_p}]$ to route the messages from u to vertices in the "bad" group V_{i_p}. The edge $(u, u_{j_{m+1}})$ is used to route messages from u to V_k, i.e., we label the edge $(u, u_{j_{m+1}})$ by the interval $[\ell(u_k), n]$.

This completes the description of the interval routing scheme. In the following we illustrate the idea further by a simple example. In Figure 2, we show a graph on 9 vertices with a $\frac{1}{2}$-threshold dominating clique of size 5. The numbers on the vertices denote the vertex label assigned according to the method discussed in Section 4.1. In Table 1, edge-labeling functions are listed of several representative vertices.

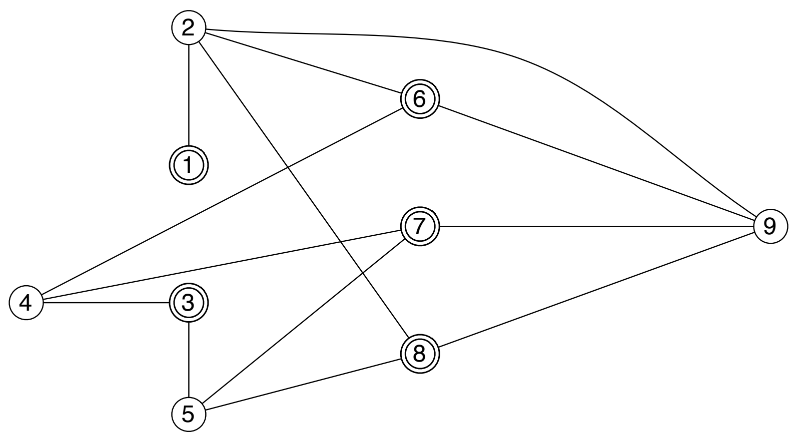

Fig. 2. An illustration of the interval routing scheme. Three groups in the partitions: $\{1, 2\}$, $\{3, 4, 5\}$, and $\{6, 7, 8, 9\}$. Vertex set $\{1, 3, 6, 7, 8\}$ is a 1/2-threshold dominating clique. Edges between vertices in the clique are omitted.

Source Vertex	Edge Label
2	L(2, 1) = [1, 2], L(2, 6) = [3, 5], L(2, 8) = [6, 9]
4	L(4, 3) = [3, 5], L(4, 6) = [1, 2], L(4, 7) = [6, 9]
9	L(9, 2) = [1, 2], L(9, 6) = [3, 5], L(9, 7) = [6, 9]

Table 1
Edge labels of outgoing links from representative source vertices. For each source vertex, outgoing links with an empty interval label are omitted.

4.3 Correctness of the Routing Scheme

In this subsection, we show that the proposed routing scheme satisfies the conditions in Definition 4.1 and has an additive stretch 1.

Let $\{V_i = \{u_i\} \cup U_i, 1 \leq i \leq k\}$ be the partition given in Lemma 4.1. Consider the vertex-labeling function $\ell(\cdot)$ and the edge-labeling function $L(\cdot, \cdot)$ defined in Sections 4.1 and 4.2.

4.3.1 The Pair $(\ell(\cdot), L(\cdot, \cdot))$ is an Interval Routing Scheme

We focus on a brief discussion on Conditions 2 and 3 in Definition 4.1. Condition 4 in Definition 4.1 is handled in the next subsection when we discuss the length of the routing paths. We consider edge labels with the source vertex in V_D and $V \setminus V_D$ separately:

(1) **Vertices in V_D.** For any vertex u_i, from the construction in Section 4.2.1, we can see that $\{L(u_i, u_j), j \neq i\} \cup \{L(u_i, v), v \in U_i\} = \{1, 2, \cdots, n\}$ and that the intervals labeling the edges are disjoint.

(2) **Vertices in $V \setminus V_D$.** It can be checked that the construction given in Section 4.2.2 guarantees that the union of the intervals labeling the edges incident to a vertex $v \in V \setminus V_D$ is $\{1, \cdots, n\}$ and the intervals are disjoint.

4.3.2 The Routing Scheme $(\ell(\cdot), L(\cdot, \cdot))$ has an Additive Stretch 1

Let u and v be two vertices in the graph. We consider two cases. First assume that (u, v) is an edge in the graph. There are 3 sub-cases:

(1) **Both u and v are in $\mathbf{V_D}$.** In this case, the construction in Section 4.2.1 guarantees that the vertex label $\ell(v) \in L(u, v)$. Thus the u-to-v routing path is just $u \to v$ and is the shortest;

(2) $\mathbf{u \in V_D}$ **but** $\mathbf{v \in U_i}$ **for some i.** If u and v are in the same group, then the u-to-v routing path is $u \to v$ and is the shortest. If u and v are in different group, the routing path is $u \to u_i \to v$ as indicated in the construction in Section 4.2.1, and has an additive stretch 1;

(3) $\mathbf{u \notin V_D}$**.** If v is in the same group as u, say V_i, then the construction given in Section 4.2.1 and at the beginning of Section 4.2.2 guarantee the routing path $u \to u_i \to v$, which has an additive stretch 1; If v is in a different group V_j for some $j \neq k$, the construction given in Section 4.2.2 (A) guarantees that $\ell(v) \in L(u, v)$. Thus, the u-to-v path $u \to v$ is the shortest; If $v \in V_k$, the construction given in Section 4.2.2 (B) guarantees the u-to-v routing path $u \to u_{j_{m+1}} \to v$, which has an additive stretch 1.

Secondly, assume that (u, v) is not an edge in the graph. Note that in this case the length of the shortest u-to-v path is at least 2. We consider several sub-cases:

(1) $\mathbf{u, v \in U_i}$ **for some i.** In this case, the u-to-v routing path is $u \to u_i \to v$, which is the shortest. In the case $i = k$, the path is $u \to u_{j_{m+1}} \to v$.
(2) $\mathbf{u \in V_i}$ **and** $\mathbf{v \in V_j}$ **with** $\mathbf{i \neq j}$**.** There are several situations to consider:
   (a) $\mathbf{u = u_i}$ **for some i and** $\mathbf{v \in U_j}$ **for some** $\mathbf{j \neq k}$**.** In this case, the u-to-v routing path is $u \to u_j \to v$ and is the shortest.
   (b) $\mathbf{u = u_i}$ **for some i and** $\mathbf{v \in U_k}$**.** If this case, the routing path is $u \to u_n \to v$ as guaranteed by the construction in Section 4.2.1.
   (c) $\mathbf{u \in U_i}$ **and is adjacent to one or more vertices in** $\mathbf{V_j}, j < k$**.** In this case, the construction given in Section 4.2.2 (A) guarantees that the u-to-v routing path is $u \to w \to u_j \to v$ where w is some vertex in V_j adjacent to u. This routing path has an additive stretch 1.
   (d) $\mathbf{u \in U_i}$ **and is not adjacent to any vertex in** $\mathbf{V_j}$ **or** $\mathbf{j = k}$**.** In this case, the construction given in Section 4.2.2 (B) guarantees that the u-to-v routing path is $u \to u_{j_p} \to u_j \to v$ where u_{j_p} is the dedicated bridge vertex. The routing path has an additive stretch 1.

5 Threshold Dominating Cliques in Random Graphs

In this section, we prove Theorem 2 by establishing the threshold behavior of the property of having an α-threshold dominating clique in the random graph $G(n, p)$. We use Markov's inequality to prove the case of $p < \alpha$ and Chebyshev's inequality to prove the case of $p > p_\alpha$. In both cases, the Chernoff bound is used to estimate certain tail probabilities.

Lemma 5.1 (Chernoof Bound [10]) *Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of independent and identically distributed Bernoulli random variables with mean p. Then, for any $t > 0$,*

$$\mathbb{P}\left\{\left|\sum_{i=1}^{n} X_i - np\right| \geq t\right\} \leq 2e^{-\frac{2t^2}{n}}.$$

Definition 5.1 *A vertex set U is said to α-threshold dominate a vertex $v \notin U$ if*

$$|N(v) \cap U| > \alpha |U|.$$

A vertex set U is an α-threshold dominating set if it α-threshold dominates every vertex $v \notin U$.

Recall that the random graph $G(n, p)$ contains each of the $\binom{n}{2}$ possible edges independently with probability p ([1]). We will be using the following notation

$DC(n,r)$: the number of the α-threshold dominating cliques of size r in $G(n,p)$.

$Q(n,r)$: the probability that a vertex set of size r is an α-threshold dominating set.

$q(n,r)$: the probability that a size-r vertex set α-threshold dominates a given vertex.

First, we use the Chernoff bound to lower bound $q(n,r)$.

Lemma 5.2 *For any random graph $G(n,p)$, the probability $q(n,r)$ satisfies*

$$q(n,r) \begin{cases} > 1 - e^{-2(p-\alpha)^2 r}, & \text{if } \alpha < p < 1 \\ < e^{-2(p-\alpha)^2 r}, & \text{if } 0 < p < \alpha \end{cases} \tag{3}$$

Proof. Let U be a subset of vertices with $|U| = r$ and let v be any vertex not in U. Then,

$$q(n,r) = \mathbb{P}\left\{\sum_{u\in U} I_u(v) > \alpha r\right\} \tag{4}$$

where $I_u(v) = 1$ if (u,v) is an edge, and $I_u(v) = 0$ otherwise. For the random graph $G(n,p)$, the variables $\{I_u(v), u \in U\}$ are mutually independent Bernoulli random variables with mean p, so that

$$\mathbb{E}\left[\sum_{u\in U} I_u(v)\right] = p|U| = pr.$$

If $\alpha < p < 1$, we have by the Chernoof bound

$$\begin{aligned} q(n,r) &= \mathbb{P}\left\{\sum_{u\in U} I_u(v) > \alpha r\right\} \\ &= 1 - \mathbb{P}\left\{\sum_{u\in U} I_u(v) \le \alpha r\right\} \\ &= 1 - \mathbb{P}\left\{\sum_{u\in U} I_u(v) \le pr - (pr - \alpha r)\right\} \\ &\ge 1 - e^{-\frac{2(pr-\alpha r)^2}{r}} \\ &= 1 - e^{-2(p-\alpha)^2 r}. \end{aligned} \tag{5}$$

Similarly, but from the opposite direction, we have if $0 < p < \alpha$,

$$
\begin{aligned}
q(n,r) &= \mathbb{P}\left\{\sum_{u\in U} I_u(v) > \alpha r\right\} \leq \mathbb{P}\left\{\sum_{u\in U} I_u(v) \geq \alpha r\right\} \\
&= 1 - \mathbb{P}\left\{\sum_{u\in U} I_u(v) < \alpha r\right\} \\
&= 1 - \mathbb{P}\left\{\sum_{u\in U} I_u(v) < pr + (\alpha r - pr)\right\} \\
&< 1 - (1 - e^{-\frac{2(pr-\alpha r)^2}{r}}) \\
&= e^{-2(p-\alpha)^2 r}. \qquad (6)
\end{aligned}
$$

■

The next lemma shows that Equation (2) in Theorem 2 always has a unique solution so that the value p_α is well-defined.

Lemma 5.3 *The equation*

$$4(p-\alpha)^2 \log_{\frac{1}{p}} e = 1 \qquad (7)$$

has a unique solution p_α in the real-valued interval $[\alpha, 1]$. Furthermore, for any $p > p_\alpha$, we have

$$4(p-\alpha)^2 \log_{\frac{1}{p}} e > 1.$$

Proof. Consider the function $f(p) = 4(p-\alpha)^2 \log_{\frac{1}{p}} e - 1$. Since $f(\alpha) = -1$ and $\lim_{p\to 1} f(p) = +\infty$, it is sufficient to show that $f(p)$ is monotonically increasing. This is true because the derivative of $f(p)$

$$f^{'}(p) = 4(p-\alpha)\frac{1}{\log p}(\frac{p-\alpha}{p\log p} - 2)$$

is positive for any $\alpha < p < 1$. ■

The next lemma shows that for a certain r and sufficiently large n, the probability $Q(n,r)$ is lower bounded by a positive constant. This fact will be used when we apply Chebyshev's inequality.

Lemma 5.4 *For any given $p > p_\alpha$, there exists a constant $Q^* > 0$ and a sufficiently small number $\epsilon = \epsilon(p) > 0$ such that for $r = (2-\epsilon)\log_{\frac{1}{p}} n$,*

$$Q(n,r) > Q^*$$

for sufficiently large n.

Proof. Recall that $Q(n,r)$ is the probability that a subset of vertices of size r is an α-threshold dominating set, and that $q(n,r)$ is the probability that a subset of vertices of size r α-threshold dominates a vertex v. We have by the independence of the edges in the random graph $G(n,p)$ that

$$Q(n,r) = (q(n,r))^{n-r}.$$

It follows from Lemma 5.2 that

$$Q(n,r) > \left(1 - e^{-2(p-\alpha)^2 r}\right)^{n-r}. \tag{8}$$

Since $r = (2-\epsilon)\log_{\frac{1}{p}} n$, we have

$$e^{-2(p-\alpha)^2 r} = e^{-2(p-\alpha)^2(2-\epsilon)\log_{\frac{1}{p}} n} = n^{-2(2-\epsilon)(p-\alpha)^2 \log_{\frac{1}{p}} e}.$$

To simply the notation, let $g(p) = -2(2-\epsilon)(p-\alpha)^2 \log_{\frac{1}{p}} e$ and rewrite the right-hand-side of the inequality (8) as

$$\begin{aligned}
\left(1 - e^{-2(p-\alpha)^2 r}\right)^{n-r} &= \left(1 - n^{g(p)}\right)^{n-r} \\
&= \left(1 - n^{g(p)}\right)^{n^{-g(p)} n^{g(p)} (n-r)} \\
&\geq \left(1 - n^{g(p)}\right)^{n^{-g(p)} n^{g(p)} n} \\
&= \left(\left(1 - n^{g(p)}\right)^{n^{-g(p)}}\right)^{n^{f(p)}}
\end{aligned}$$

where $f(p) = 1 + g(p) = 1 - 2(2-\epsilon)(p-\alpha)^2 \log_{\frac{1}{p}} e$.

From Lemma 5.3, we have that $1 - 4(p-\alpha)^2 \log_{\frac{1}{p}} e$ is strictly less than 0 for any given $p > p_\alpha$. It follows that there is a sufficiently small (but still positive) number $\epsilon = \epsilon(p)$ such that $f(p) = 1 - 4(p-\alpha)^2 \log_{\frac{1}{p}} e + 2\epsilon(p-\alpha)^2 \log_{\frac{1}{p}} e < 0$, and consequently $n^{f(p)} \to 0$. Since

$$\lim_n \left(1 - n^{g(p)}\right)^{n^{-g(p)}} = e^{-1} > 0,$$

we see that there exists a constant Q^* such that for sufficiently large n, $Q(n,r) > Q^* > 0$. The lemma is proved. ■

We now show that for any given $p > p_\alpha$, $G(n,p)$ has an α-threshold dominating clique with high probability, thereby establishing the first part of Theorem 2.

Proposition 5.1 *Consider $DC(n,r)$, the number of α-threshold dominating cliques in $G(n,p)$. Let $p > p_\alpha$ and $r = (2-\epsilon)\log_{\frac{1}{p}} n$ where $\epsilon = \epsilon(p) > 0$ is a* *small constant specified in Lemma 5.4. We have*

$$\lim_{n} \mathbb{P}\left\{DC(n,r)=0\right\}=0. \tag{9}$$

Proof. The expected number of α-threshold dominating cliques is

$$\mathbb{E}\left[DC(n,r)\right]=\binom{n}{r}p^{\binom{r}{2}}Q(n,r) \tag{10}$$

Therefore by Lemmas 5.2 and 5.4, we have for sufficiently large n

$$\mathbb{E}\left[DC(n,r)\right]\geq \binom{n}{r}p^{\binom{r}{2}}Q^{*}.$$

By Stirling's formula, the right-hand-side of the above is asymptotically lower bounded by

$$\frac{1}{\sqrt{2\pi r}}\left(\frac{en^{\frac{\epsilon}{2}}}{\sqrt{p}r}\right)^{r}Q^{*}$$

which tends to ∞ as $n\to\infty$

By Chebyshev's inequality, we have

$$\begin{aligned}
\mathbb{P}\left\{DC(n,r)=0\right\} &\leq \mathbb{P}\left\{\left|DC(n,r)-\mathbb{E}\left[DC(n,r)\right]\right|\geq \mathbb{E}\left[DC(n,r)\right]\right\}\\
&\leq \frac{\mathbb{E}\left[\left(DC(n,r)-\mathbb{E}\left[DC(n,r)\right]\right)^{2}\right]}{\left(\mathbb{E}\left[DC(n,r)\right]\right)^{2}}\\
&=\frac{\mathbb{E}\left[DC^{2}(n,r)\right]}{\left(\mathbb{E}\left[DC(n,r)\right]\right)^{2}}-1.
\end{aligned}$$

To estimate the right-hand-side of the above inequality, we note that $DC(n,r)$ can be written as

$$DC(n,r)=\sum_{U}I_{U}$$

where the sum is over the collection of vertex subsets of size r and I_U is the indicator function that U is an α-threshold dominating clique in $G(n,p)$. It follows that

$$\mathbb{E}\left[DC^{2}(n,r)\right]=\sum_{(U,V)}\mathbb{E}\left[I_{U}I_{V}\right]$$

where the sum is over all the (ordered) pairs of vertex sets of size r. By grouping the terms in the above summation according to the number of vertices that a pair of vertex sets (U,V) have in common, we get an expression of $\mathbb{E}\left[DC^{2}(n,r)\right]$ as follows. Using the notation $\binom{0}{2}=\binom{1}{2}=0$ to accommodate the special cases of $l=0$ and $l=1$, the expectation of $DC^{2}(n,r)$ cam be represented as

$$\mathbb{E}\left[DC^2(n,r)\right]=\sum_{l=0}^{r}\binom{n}{r}\binom{r}{l}\binom{n-r}{r-l}p^{2\binom{r}{2}-\binom{l}{2}}P(l)$$

where

(1) $\binom{n}{r}\binom{r}{l}\binom{n-r}{r-l}$ is the total number of pairs of vertex subsets of size r that have l vertices in common,
(2) $p^{2\binom{r}{2}-\binom{l}{2}}$ is the probability that such a pair of vertex sets both induce a clique, and
(3) $P(l)$ is the conditional probability that such a pair of vertex subsets are both α-threshold dominating sets given that they both induce a clique.

From Equation (10),

$$(\mathbb{E}\left[DC(n,r)\right])^2=\left(\binom{n}{r}p^{\binom{r}{2}}Q(n,r)\right)^2=\binom{n}{r}^2p^{2\binom{r}{2}}Q^2(n,r).$$

We write $\frac{\mathbb{E}\left[DC^2(n,r)\right]}{(\mathbb{E}[DC(n,r)])^2}$ as

$$\frac{\mathbb{E}\left[DC^2(n,r)\right]}{(\mathbb{E}\left[DC(n,r)\right])^2}=\frac{\sum\limits_{l=0}^{r}\binom{n}{r}\binom{r}{l}\binom{n-r}{r-l}p^{2\binom{r}{2}-\binom{l}{2}}P(l)}{\binom{n}{r}^2p^{2\binom{r}{2}}Q^2(n,r)}=a_n+b_n,$$

where

$$\begin{aligned}a_n&=\frac{\sum\limits_{l=0}^{1}\binom{n}{r}\binom{r}{l}\binom{n-r}{r-l}p^{2\binom{r}{2}-\binom{l}{2}}P(l)}{\binom{n}{r}^2p^{2\binom{r}{2}}Q^2(n,r)}\\&=\binom{n}{r}^{-1}\binom{n-r}{r}\frac{P(0)}{Q^2(n,r)}+\binom{n}{r}^{-1}r\binom{n-r}{r-1}\frac{P(1)}{Q^2(n,r)}\end{aligned}$$

and

$$b_n=\sum_{l=2}^{r}\binom{n}{r}^{-1}\binom{r}{l}\binom{n-r}{r-l}p^{-\binom{l}{2}}\frac{P(l)}{Q^2(n,r)}.$$

We claim that $\lim\limits_{n} b_n=0$. To prove the claim, first note that

$$\begin{aligned}
&\binom{n}{r}^{-1}\binom{r}{l}\binom{n-r}{r-l}\\
&\le r^l \frac{r!}{n(n-1)\cdots(n-r+1)} \frac{(n-r)(n-r-1)\cdots(n-2r+l+1)}{(r-l)!}\\
&\le r^l \frac{r(r-1)\cdots(r-l+1)}{n(n-1)\cdots(n-l+1)} \qquad \text{(note that } l \le r\text{)}\\
&\le \frac{r^{2l}}{(n-l+1)^l}.
\end{aligned}$$

By Lemma 5.4, $Q(n,r)$ is lower bounded by a constant Q^* for sufficiently large n. Taking into consideration the fact that $P(l) \le 1$, we have for the constant $C^* = (\frac{1}{Q^*})^2$ and sufficiently large n,

$$\begin{aligned}
b_n &\le C^* \sum_{l=2}^{r} \binom{n}{r}^{-1}\binom{r}{l}\binom{n-r}{r-l} p^{-\binom{l}{2}}\\
&\le C^* \sum_{l=2}^{r} \frac{r^{2l}}{(n-l+1)^l} p^{-\binom{l}{2}}\\
&= C^* \sum_{l=2}^{r} \left(\frac{r^2}{(n-l+1)p^{(l-1)/2}}\right)^l\\
&\le C^* \sum_{l=2}^{r} \left(\frac{r^2}{(n-l+1)p^{(r-1)/2}}\right)^l \qquad \text{(since } p \le 1\text{)}\\
&= C^* \sum_{l=2}^{r} \left(\frac{p^{1/2}r^2}{(n-l+1)n^{-1+\epsilon/2}}\right)^l \qquad \text{(recall that } r = (2-\epsilon)\log_{1/p} n\text{)}\\
&= C^* \sum_{l=2}^{r} \left(\frac{p^{1/2}r^2}{(1-(l-1)/n)n^{\epsilon/2}}\right)^l\\
&\le C^* \sum_{l=2}^{r} \left(\frac{2p^{1/2}r^2}{n^{\epsilon/2}}\right)^l.
\end{aligned}$$

Noticing that $\frac{2p^{1/2}r^2}{n^{\epsilon/2}} \to 0$ and is independent of l, we see the claim that $\lim_n b_n = 0$ holds.

Since $\lim_n \binom{n}{r}^{-1} r\binom{n-r}{r-1} = 0$ and $Q(n,r)$ is lower bounded by a constant (Lemma 5.4), we see that the second term of a_n also tends to 0, i.e.,

$$\lim_n \binom{n}{r}^{-1} r \binom{n-r}{r-1} \frac{P(1)}{Q^2(n,r)} = 0.$$

To prove the theorem, it is thus sufficient to show that

$$\lim_n \frac{P(0)}{Q^2(n,r)} = 1 \tag{11}$$

Let U and W be two vertex sets of size r such that $|U \cap W| = 0$. Consider the conditional probability $P(0)$ that W is an α-threshold dominating set given that U is an α-threshold dominating set. Let E_U (or E_W) be the event that U (respectively W) is an α-threshold dominating set. We have

$$P(0) = \mathbb{P}\{E_U\}\, \mathbb{P}\{E_W | E_U\} = Q(n,r)\mathbb{P}\{E_W|E_U\}.$$

So, it is sufficient to consider the ratio

$$\frac{\mathbb{P}\{E_W|E_U\}}{Q(n,r)}.$$

Recall that $Q(n,r) = (q(n,r))^{n-r}$. Let E_W^u be the event that the vertex u is α-threshold dominated by W. By the independence of the edges in the random graph $G(n,p)$, we have

$$\begin{aligned}\frac{\mathbb{P}\{E_W|E_U\}}{Q(n,r)} &= \frac{(q(n,r))^{n-r-r} \prod_{u \in U} \mathbb{P}\{E_W^u|E_U\}}{(q(n,r))^{n-r}} \\ &= \frac{\prod_{u \in U} \mathbb{P}\{E_W^u|E_U\}}{(q(n,r))^r}.\end{aligned}$$

Since for any vertex u in U, knowing that U is an α-threshold dominating set increases the likelihood for u to be dominated by W, we have

$$\mathbb{P}\{E_W^u|E_U\} > \mathbb{P}\{E_W^u\}.$$

Thus, we have

$$1 \leq \frac{\prod_{u \in U} \mathbb{P}\{E_W^u|E_U\}}{(q(n,r))^r} < \frac{1}{(q(n,r))^r}.$$

For $r = (2-\epsilon)\log n$ and $p > p_\alpha$, we have by Lemma 5.2 that

$$\begin{aligned}\lim_n \frac{1}{(q(n,r))^r} &\leq \lim_n \left(1 - e^{-2(p-\alpha)^2 r}\right)^{-r} \\ &= \lim_n \left(1 - e^{-2(p-\alpha)^2 r}\right)^{e^{2(p-\alpha)^2 r}(-re^{-2(p-\alpha)^2 r})} \\ &= 1,\end{aligned}$$

where the last equation is due to the fact that $\lim_n(-re^{-2(p-\alpha)^2 r}) = 0$. This proves the equation (11). The proposition follows. ■

The following proposition proves the second part of Theorem 2.

Proposition 5.2 *For any $p < \alpha$, we have*

$$\lim_n \mathbb{P}\{G(n,p) \text{ has an } \alpha\text{-threshold dominating clique}\} = 0.$$

Proof. Consider the expected number of α-threshold dominating cliques

$$\mathbb{E}\left[\sum_{r=1}^{n} DC(n,r)\right] = \sum_{r>2\log_{\frac{1}{p}} n} \binom{n}{r} p^{\binom{r}{2}} Q(n,r) + \sum_{r\le 2\log_{\frac{1}{p}} n} \binom{n}{r} p^{\binom{r}{2}} Q(n,r). \quad (12)$$

First, we estimate the first term on the right-hand-side of the above equation (12). Based on the bound $\binom{n}{r} \le (\frac{en}{r})^r$, we have

$$\begin{aligned}
\sum_{r>2\log_{\frac{1}{p}} n} \binom{n}{r} p^{\binom{r}{2}} Q(n,r) &\le \sum_{r>2\log_{\frac{1}{p}} n} (\frac{en}{r})^r p^{\binom{r}{2}} \qquad \text{(since } Q(n,r) \le 1\text{)} \\
&= \sum_{r>2\log_{\frac{1}{p}} n} \left(\frac{enp^{r/2}}{rp^{1/2}}\right)^r \\
&\le \sum_{r>2\log_{\frac{1}{p}} n} \left(\frac{e}{rp^{1/2}}\right)^r,
\end{aligned}$$

where the last inequality is because for any $r \ge 2\log_{1/p} n$, we have

$$np^{r/2} \le np^{\log_{1/p} n} \le 1.$$

Since $\frac{e}{rp^{1/2}} < 1$ for sufficiently large n, it follows that

$$\begin{aligned}
\lim_n \sum_{r>2\log_{\frac{1}{p}} n} \binom{n}{r} p^{\binom{r}{2}} Q(n,r) &\le \lim_n \sum_{r>2\log_{\frac{1}{p}} n} \left(\frac{e}{rp^{1/2}}\right)^r \\
&\le \lim_n n \left(\frac{e}{2p^{1/2}\log_{1/p} n}\right)^{2\log_{1/p} n} \\
&= 0.
\end{aligned}$$

The second term on the right-hand-side of (12) also tends to 0 since from the case of $p < \alpha$ in Lemma 5.2 the probability $Q(n,r)$ is exponentially small.

Recall that $\sum_{r=1}^{n} DC(n,r)$ is the total number of threshold dominating cliques in a random graph. The proposition follows from Markov's inequality:

$$\mathbb{P}\left\{\sum_{r=1}^{n} DC(n,r) > 0\right\} \leq \mathbb{E}\left[\sum_{r=1}^{n} DC(n,r)\right].$$

$\blacksquare$

This completes the proof of Theorem 2. The proof of Theorem 3 is similar to the proof of Proposition 5.1 and Proposition 5.2. The only difference is that instead of using the Chernoff bound to estimate the probability $q(n,r)$ and $Q(n,r)$, we can use explicit expressions in the proof of Theorem 3. Consider a vertex set V_β of size r and a given vertex $v \notin V_\beta$. The number of vertices in V_β that are adjacent to v is a binomial random variable with parameters p and r. It follows that the probability for v to have more than β neighbors in V_β is

$$1 - \sum_{i=0}^{\beta} \binom{r}{i} p^i (1-p)^{r-i}.$$

Let $Q(n,r,\beta)$ be the probability that a vertex set V_β of size r is such that for any $v \notin V_\beta$, $|N(v) \cap V_\beta| > \beta$. Due to the independence of the edges in a random graph, we have

$$\begin{aligned} Q(n,r,\beta) &= \left(1 - \sum_{i=0}^{\beta} \binom{r}{i} p^i (1-p)^{r-i}\right)^{n-r} \\ &= \left(1 - (1-p)^r \sum_{i=0}^{\beta} \binom{r}{i} p^i (1-p)^{-i}\right)^{n-r}. \end{aligned} \tag{13}$$

Since β is a fixed constant and $r \in O(\log n)$, the summation term in the above is in $O(poly(\log n))$.

Let $DC(n,r,\beta)$ be the number of size-r vertex sets that induce a clique and dominate the graph in the above sense. The expected number of cliques dominating the graph in the above sense is

$$\sum_{r=1}^{n} \mathbb{E}\left[DC(n,r,\beta)\right] = \sum_{r=1}^{n} \binom{n}{r} p^{\binom{r}{2}} Q(n,r,\beta).$$

It can be shown that

$$\lim_{n} \sum_{r=1}^{n} \mathbb{E}\left[DC(n,r,\beta)\right] = \begin{cases} 0, \text{ if } \log_{\frac{1}{p}}(\frac{1}{p}-1) > \frac{1}{2} \\ 1, \text{ if } \log_{\frac{1}{p}}(\frac{1}{p}-1) < \frac{1}{2}. \end{cases} \tag{14}$$

We omit the proof of of Equation (14) for general constant $\beta > 1$ since it is almost identical to the proof for the special case of $\beta = 1$ detailed in [2]. This is due to our earlier observation that the summation term in Equation (13) is in $O(poly(\log n))$ and consequently has no impact on the asymptotical behavior.

Solving the equation $\log_{\frac{1}{p}}(\frac{1}{p} - 1) = \frac{1}{2}$ gives us the threshold value $\frac{3-\sqrt{5}}{2}$. Similarly to the proof of Propositions 5.1 and 5.2, Theorem 3 can be proved.

6 Conclusions

There is a gap between the lower bound and the upper bound on the probability for the existence of an α-threshold dominating clique in the random graph $G(n, p)$. Closing this gap is an interesting future work.

We remark on the complexity of the threshold dominating clique problem. The problem of finding a dominating clique is NP-complete and fixed-parameter intractable as well (See, e.g., Appendix 1 of [4]). The problem of finding a dominating clique such that each vertex is dominated by more than $\beta > 0$ (a fixed constant), as the case dealt with in Theorem 3, can also be shown to be NP-complete and fixed-parameter intractable by a reduction from the threshold dominating set problem (see, e.g., [3]). The complexity of the α-threshold dominating clique problem defined in the current paper is not clear.

At least two exact (worst-case exponential) algorithms for the dominating clique problem have been proposed for general graphs [2,9], both of which are based on branch-and-reduce and backtracking. The one in [9] is shown to have a worst-case running time $O(1.339^n)$ where n is the number of the vertices in a graph. The one in [2] is designed with special considerations on features of random graphs and has been implemented and empirically shown to perform quite well for random graphs of size up to 1000 vertices. Customizing the latter to solve the α-threshold dominating clique problem is not hard. More more recently, we have been able to show that for any random graph $G(n, p)$ with $p > \frac{3-\sqrt{5}}{2}$ and any constant β a simple greedy algorithm finds with high probability a clique U such that any vertex $v \notin U$ has at least β neighbors in U ([6]). The analysis of the greedy algorithm, however, does not apply to the problem of finding α-threshold dominating cliques.

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